Investment Timing for Dynamic Business Expansion

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Abstract

In a dynamic setting, when growth investments are proportional to firm size, we derive the return on capital (ROC) required for value creating expansion investments. The endogenously determined cost of capital uniformly exceeds this rate. A manager accepts marginal investments that have ROC less than the cost of capital because they facilitate larger and more valuable future investments when earnings improve. This result means that managerial application of the cost of capital as a hurdle rate for growth investment or as a benchmark for corporate performance is improperly conservative. Further, this result is opposite to investment deferral that the investment under uncertainty literature recommends due to investment irreversibility.

Keywords: Corporate Performance, Growth, Business Expansion.

An enduring legacy of Modigliani and Miller (1958, 1963) is the common managerial practice of using a firm's cost of capital for corporate performance evaluation and for business expansion analysis when expansion has the same risk as existing operations. Textbooks on corporate finance advance this methodology and it has been modernized and promoted for corporate practice by academics, using, for example, residual income measurement, Kaplan (1982), and EVA analysis,[1] Stewart (1991). Countless recent articles and popular press books, for example, Copeland, Koller, and Murrin (1991), Ehrbar(1998), and Grant and Abate (2001), promote the virtues of this mode of corporate performance measurement for capital allocation, strategic planning, and even portfolio decisions. When firms tie these performance measures to managerial compensation, Rogerson (1997), Reichelstein (1997), and Christensen, Feltham, Wu (2002) show that they encourage an "economic" use of capital by managers.

For proportional growth investments, we derive the value maximizing return on capital (ROC) required for expansion investment, which is also the rate that investors and firms must apply against capital to measure economic profit in corporate performance evaluation. We prove that the endogenously determined cost of capital uniformly exceeds the value maximizing ROC expansion boundary. This result means that managerial application of the cost of capital as a hurdle rate for expansion or as a benchmark for corporate performance is unduly conservative.

There are two reasons that the cost of capital uniformly exceeds the dynamic expansion boundary. First, a manager may accept a marginal investment today to increase future

[1] EVA stands for Economic Value Added and is a registered trademark of Stern Stewart Inc.

capital. Current investment allows larger future investment because it, like the current investment, is proportional to capital. Future investment has greater dollar value for a firm made larger by today's investment. Scaling up the firm today, even if ROC on current investment is below the cost of capital, increases the dollar value of future growth investment when profitability stochastically improves. Second, the fixed but avoidable capital expenditures of business expansion create a growth leverage risk that we endogenize in the cost of capital. Growth leverage increases the cost of capital. Because proportional growth investment decreases the expansion boundary, but increases the cost of capital, the cost of capital uniformly exceeds the dynamic expansion boundary.

Our results are opposite to investment deferral for solitary investments in McDonald and Siegel (1986) and Dixit and Pindyck (1994). Downside earnings risk for a one-time irreversible investment is an essential feature of the study of investment under uncertainty and the investment deferral option. The possibility that earnings for an irreversible investment might disappoint leads a manager to defer the start of a new investment. They show that the profit boundary to exercise a "start-option" on a solitary new investment *exceeds* the profit boundary for zero net present value (NPV) of the underlying investment. The start decision (equivalently, the option to defer the investment), is a call option on the underlying investment and the manager exercises it only when the investment is "in the money" and has positive rather than zero NPV. Equivalently, the return boundary for starting the investment *exceeds* the investment's cost of capital once undertaken.

There are a number of economic factors that influence a manager's incentive to defer an irreversible investment. These factors include: partial reversibility,[2] sequences of growth investments,[3] investment learning,[4] strategic investments,[5] capital stock adjustment costs,[6] and agency costs.[7] While these factors can reduce the value of the deferral option and reduce the manager's incentive for investment delay, there is yet no research that demonstrates that these factors, in isolation or in combination, in a dynamic setting, can cause the deferral option to reverse, become negative, and induce investment earlier than recommended by Modigliani and Miller's (1958, 1963) conventional cost of capital analysis.

We extend the existing literature by demonstrating this possibility in this paper. While the modeling assumptions we employ are not markedly different from the literature we review below, the primary result we derive in this paper is both novel and important to practicing managers. We show that if a firm has a sequence of growing growth investments, limited from above to a proportion of capital (maximum growth) as well as below (irreversibility), then the dynamic expansion boundary is uniformly less than the endogenously determined cost of capital. Because of the intuitive appeal of deferral for one-time irreversible investments in McDonald and Siegel (1986) and Dixit and Pindyck (1994), investment

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[2] See, for example, Kandel and Pearson (2002).

[3] See, for example, Pindyck (1988) and Abel and Eberly (1994).

[4] See, for example, Bergemann and Hege (1998) and Grenadier and Weiss (1997).

[5] See, for example, with the related literature review, Cottrell and Sick (2002). A manager undertakes strategic investments that preempt competitive entry earlier than absent this competition. However, no one has shown that this investment is sooner in a real options setting compared to standard cost of capital analysis in the *same* strategic environment. Cottrel and Sick compare the attractions of follower to first mover strategies.

[6] See, for example, Abel and Eberly (1994) and Fisher, Carlson, and Giammarino (2004).

[7] See, for example, Christensen and Feltham (2002) and Bergemann and Hege (1998).

timing and deferral has not been studied as actively in the investment under uncertainly literature as it should be. We bridge this research gap in this paper.

Two of the above determinants of the deferral option's value are particularly important for our research: sequences of growth investments and capital stock adjustment costs. The possibility of value enhancing expansion investments regardless of whether future earnings exceed expectations or not makes investment deferral less imperative for the manager of an investment that has a sequence of growth options compared to a solitary investment. If earnings disappoint, then, nonetheless, once earnings improve the manager can make supplementary value enhancing expansion investments that cannot be made with a solitary investment. Similarly, if earnings exceed expectations, then, the manager can also make value enhancing expansion investments that cannot be made with a solitary investment. Upside earnings potential is greater and downside earnings risk is no greater for an investment with subsequent expansion opportunities compared to an otherwise equivalent one-time investment. Greater upside earnings potential reduces the value of the deferral option for investments with subsequent expansion opportunities.

For particular parameter values, models of sequential growth investment in Dixit and Pindyck (1994, pp 371-372), Abel (1983), Caballero (1991) and Aguerrevere (2003) find that, contrary to solitary investments in McDonald and Siegel (1986) and Dixit and Pindyck (1994), volatility can encourage investment despite investment irreversibility. That is, profit thresholds for value maximizing investment can decrease with profit volatility. This result illustrates that models of sequential growth and expansion investment have distinct characteristics from solitary investments. In the current paper, we show that when growth investment is proportional to firm size, volatility always encourages growth investment regardless of model parameters. More importantly, we add to the set of differences that exist between solitary investments and sequential growth investments by demonstrating a result that not only does not appear in prior research, but is also important to practicing business managers. The endogenously determined cost of capital uniformly exceeds the value maximizing ROC expansion boundary, and therefore, a value maximizing manager expands a business sooner than a manager who relies on cost of capital as an expansion hurdle rate. This result exists regardless of whether we impose the cost of capital exogenously, like the above research, or determine it endogenously.

It is important to compare value maximizing models with cost of capital analysis because the cost of capital is ingrained in the corporate practice of business expansion evaluation. As managers recognize that the cost of capital is not an appropriate benchmark for business investment decisions in dynamic settings in the presence of various real options, they will look to the financial literature for guidance. At the current time, the most likely source of guidance is the investment under uncertainty literature, for example, Dixit and Pindyck (1994). Managers will discover that the predominant modeling feature of this literature is the investment deferral option. Managers exercise their "call option" on the value of an underlying investment when downside value protect no longer exceeds forgone profit from investment deferral. This deferral means that the investment return boundary exceeds the investment's cost of capital. Because the cost of capital is relatively easy to measure, in following the existing investment under uncertainty literature, managers will add a positive factor to the cost of capital to reflect the deferral option when calculating the expansion boundary for their businesses. We show that this conjectured managerial heuristic for business expansion is inappropriately conservative.

Any intertemporal investment model with growth options requires some limit on investment spending. Pindyck (1988) presumes that investment has diminishing returns for profit creation, which lessens the appeal of large investments. Abel and Eberly (1994), Zhang (2005), Fisher, Carlson, and Giammarino (2004) and Cooper (2006) presume capital stock adjustment costs that compel a manager to defer what otherwise might be attractive investments.

We limit investment spending to a constant fraction of capital per unit time, which implies that capital stock adjustment costs are zero for capital adjustments below this limit and infinitely great thereafter. One can interpret limited per period investment in differ ways depending upon the firm under investigation. However, a commonality for all firms, for example, is constrained managerial talent. Investment is limited by the fact that the firm can train new managers only at a rate proportional to the pool of existing managers. This presumption is realistic because corporations make even large investments over a period of development and implementation. Limited proportional investment is a feature of the constant growth version of the discounted dividend model for common share valuation taught ubiquitously to introductory students of finance– the Gordon growth model – popularized by Myron J. Gordon (see, Gordon and Shapiro (1956)). In the Gordon model, if a firm has a return on investment greater than the cost of capital, the manager undertakes positive NPV growth investments limited to a fraction "g" of capital per time-period. Securities analysts commonly use constant growth models and variants (like, for example, two and three growth stages) for investment decisions. See, for example, chapter 5 of Damodaran (1994) for a discussion of corporate determinants of growth. Our research is the dynamic extension of the Gordon growth model.

Our paper contributes to a recent growing literature on risk measurement in dynamic models of managerial choice, for example, Berk, Green, and Naik (1999), Zhang (2005), Goldstein, Ju, and Leland (2001), Carlson, Fisher, and Giammarino (2004), and Cooper (2006). Berk, Green, and Naik (1999) investigate corporate growth options. Because these opportunities are of heterogeneous risk, their model is not suited to expansion analysis. Goldstein, Ju, and Leland (2001) study optimal bankruptcy in a dynamic setting, but they do not consider capital growth. Because spontaneous earnings growth is unrealistic, we presume, alternatively, that earnings growth depends on capital growth. Zhang (2005) and Cooper (2006) use real options models with dynamic risk adjustment for idle physical production capacity in economic downturns to explain relations between equity returns and the book to market ratio. They do not consider the relation between the value maximizing profit boundary and the cost of capital. Carlson, Fisher, and Giammarino (2004) study the impact of corporate operating leverage and growth opportunities on asset returns. Their model provides a rational explanation for both size and book to market effects in the cross-section of asset returns. However, it offers neither closed form solutions nor analytic results in a general context nor a comparison of the optimal investment policy with the cost of capital.

Because we make no assumption that is distinctly novel, we expect that special cases or modifications to models in this literature exhibit the principal result that we derive in the current paper. This possibility notwithstanding, none of the above research demonstrates, as we do in this paper, that a value maximizing manager temporally accelerates expansion investment compared to a manager who makes expansion investments with the cost of capital as a hurdle rate. We choose not to study an existing model because, for expediency and tractability, much of this literature relies on numerical analysis and simulation for results. We choose a simple representation for capital stock adjustment costs, which we, nonetheless, believe is realistic, and therefore, we derive a closed form solution for the value maximizing ROC expansion boundary and we analytically prove that when growth investments are proportional to firm size, in a dynamic setting, the cost of capital is an unduly conservative benchmark for expansion analysis.

In Section I, we model business expansion. In Section II, we derive the value maximizing expansion boundary and compare it to the cost of capital. In Section III, we conclude with a summary.

I. Dynamic Versus Static Business Expansion

A. Operating Profit

In this subsection, we describe the stochastic process for operating profit[8] that arises from a combination of a firm's existing operations and growth investments. Let X_t represent annual dollar operating profit at time *t*, which grows at the rate *g* when the manager expands the business, and zero, otherwise,

$$\frac{dX}{X} = \begin{cases} \sigma dz, & no\ \ growth \\ \\ gdt + \sigma dz, & growth \end{cases} \tag{1}$$

[8] Of course, "operating profit" is a cash flow based measure.

where dz is a standard Gauss-Weiner process.

Because operating profit, X, is strictly positive, the manager neither abandons nor suspends existing operations. The business generates instantaneous operating profit, Xdt, regardless of whether the manager invests for expansion or not. Strictly positive operating income is characteristic of large industrial and commercial firms with existing operations, but not characteristic of small (often) development oriented firms. While creating commercially viable products, "start-up" firms bear not only development risk, but also negative operating income before they create markets and earn revenues. We study the expansion of firms with existing operations rather than initial "start-up" investments.

B. Growth Investments

Because growth is neither spontaneous nor unplanned, the manager invests to create profit growth. In free cash flow valuation, asset value depends on predicted future free cash flow, part of which is the required expenditure to generate profit growth. We presume this expenditure depends on existing asset replacement cost, which we represent with, B, capital stock. Monitoring capital is relevant because of this dependence. An investment, $gBdt$, grows expected operating profit at the annual rate g and increases capital to $B(1+gdt)$.

We presume that the manager makes expansion investments over dt of either zero or $gBdt$. Because capital stock adjustment costs are zero for an investment up to $gBdt$ and infinitely great thereafter, and because the production technology has constant returns to scale, there are no benefits or costs, proportionately, to large investments. While the manager might prefer a larger investment, the limit on investment spending is $gBdt$. Total yearly investment is a random amount between 0 and gB (uncompounded). If the manager neglects expansion for a period of time, this neglect cannot subsequently be reversed with more intense future expansion investment. Regardless of the past, the limit on investment spending over dt is $gBdt$.

Capital grows at the rate, *g*, if the manager expands the business, but it never declines: investment is irreversible. Thus,

$$\frac{dB}{B} = \begin{cases} 0, & no\ growth \\ gdt, & growth \end{cases} \tag{2}$$

Irreversible capital investment is the cost of ending non-growth for growth. There is no cost of ending growth for non-growth.

C. Return on Capital

Because we study expansion of an existing business rather than a new business activity, the efficiency of expansion equals that of the existing business in generating profit. Measure efficiency of the existing business with the ratio of profit to capital, $Y \equiv \frac{X}{B}$, the rate of return on capital (ROC). When the manager makes an expansion investment, incremental *dollar* profit in the next instant, *Δt*, is $X_0 e^{g\,\Delta t} - X_0 \approx gX_0 \Delta t$. Incremental profit divided by incremental investment, incremental ROC, is the same as ROC for existing operations, $gX_0\Delta t \big/ gB\Delta t = X_0 \big/ B = Y$. More formally, use Ito's lemma to find,

$$d\tilde{Y} = \begin{cases} \frac{\partial Y}{\partial X} dX = Y\sigma dz, & no\ growth \\ \frac{\partial Y}{\partial X} dX + \frac{\partial Y}{\partial B} dB = Y\sigma dz, & growth \end{cases} \tag{3}$$

The process for ROC, $\tilde{Y}$, depends upon neither the manager's expansion decision (the two branches of (3) equal one another) nor the proportionality constraint on investment and profit growth, g. When profit growth requires capital growth, ROC is a martingale, $E[\tilde{Y}_t] = Y_0$. ROC for a small firm is expected future ROC, regardless of the growth factor, g, when the small firm, perhaps, becomes large,[9] $E[\tilde{Y}_t] = Y_0$.

In a two-period model, Abel, Dixit, Eberly, and Pindyck (1996), show that if capital's price is expected to increase, then because future investment is more costly than current investment, the incentive to invest currently increases. The results in our model do not depend on this "price of capital" effect. Whether or not capital is pricey depends upon the profit that capital generates, ROC. We expect no change in capital's price, $E[\tilde{Y}_t] = Y_0$. Constrained capital growth neither encourages nor discourages current investment. The playing field is level for our comparison of the dynamic expansion boundary and the cost of capital.

When profit growth at the rate g necessitates capital growth at the rate g, a firm's return on investment is ROC and not ROC plus a growth factor because there is no expected ROC growth like there is for spontaneous profit growth investments. We have not left out an unaccounted value increase. An analysis of a static environment[10] is helpful to illustrate the point. With *spontaneous* profit growth, investment return, the IRR, satisfies, X/(IRR-g)-B=0, and, IRR = ROC+g. On the other hand, if profit growth at the rate g requires capital growth at the rate g, then, IRR satisfies, (X-g*B)/(IRR-g)-B=0, and, IRR=ROC regardless of the

[9] In practice, smaller firms are more likely development oriented than large firms, and large firms are more likely survivors with greater ex-post return on capital than small firms. Otherwise, we expect that there is no difference between return on capital for small versus large firms. Of course, this is an empirical question.

[10] See Gordon and Shapiro (1956) and Brealey, Myers, and Allen (2006, ch. 4) for more on the value of growth opportunities in a static environment.

growth factor, g. Further, in the static environment, if ΔB is an incremental expansion investment and ω is the cost of capital, then the NPV of this investment with subsequent future expansion investments that grow both profit and capital at the rate $g < \omega$, is[11]: $NPV=(Y\times\Delta B-g\times\Delta B)/(\omega-g)-\Delta B$. NPV is positive and a manager who relies on cost of capital benchmarking makes the incremental investment when $ROC=Y \geq \omega$, regardless of the growth factor g. Because this analysis is independent of g, if the original expansion investment is positive NPV, then the manager's expectation is that future expansion investments are also positive NPV. Any investment, including expansion investment, generates a perpetual flow of non-growing expected profit Y per dollar of added capital. The IRR satisfies $Y/IRR-1=0$, so, $IRR=Y$, and the manager makes the investment when the return, $ROC=Y$, exceeds the cost of capital, $Y\geq\omega$. This description of the manager's investment environment suggests that rather than commit indefinitely to growth investment he/she will likely follow a dynamic investment strategy in which he/she expands the business when the ROC exceeds the cost of capital, ω. This conjectured dynamic investment strategy creates growth leverage risk which increases the cost of capital above the constant, ω. In addition, in a dynamic setting, the manager must recognize capital irreversibility and that current investment relaxes the capital constraint on future growth investments when, perhaps,

[11] One might discount expected profit at a rate greater than the riskless rate and expansion investments at the riskless rate if, in the static environment, expansion investments are permanent, and thus, non-random. In this case, the ROC expansion threshold exceeds the profit discount rate. Let r^* be the profit discount rate and r be the riskless rate, $r^*\geq r$. The manager makes an incremental investment, ΔB, when $Y\geq r^*$. In addition, if the manager commits to permanent subsequent expansion when investment value with expansion exceeds investment value without expansion: $Y\times\Delta B/(r^*-g)-g\times\Delta B/(r-g)\geq Y\times\Delta B/r^*$, for $g<r$, then the ROC expansion threshold is $r^*\times (r^*-g)/(r-g)$ and exceeds the profit discount rate, r^*. In section II, we show that both these rates exceed the value maximizing ROC expansion threshold in a dynamic setting. When covariance risk is zero, even the riskless interest rate exceeds this value maximizing expansion threshold.

earnings improve. The next section investigates the manager's dynamic value maximizing investment strategy.

II. Cost of Capital and the Value Maximizing Expansion Boundary

A. Market to Capital Ratio

We use the valuation methodology of Goldstein, Ju, and Leland (2001) to find the value of a business, *V(X,B)*, that has investment and profit growth restricted to a fraction, *g*, of capital, *B*. Goldstein, Ju, and Leland (2001) model profit growth, but not capital growth required to produce this growth. In appendix A, we formally relate profit growth with capital growth through the manager's expansion decision.

The *risk-adjusted* process, X', for operating profit is:

$$\frac{dX'}{X'} = \begin{cases} -\theta\sigma_{x,c}dt + \sigma dz, & \textit{no growth} \\ \left(g - \theta\sigma_{x,c}\right)dt + \sigma dz, & \textit{growth} \end{cases} \tag{4}$$

where $\theta \geq 0$ is the coefficient of constant relative risk aversion for a representative investor, σ_{xc} is the covariance of the log of operating profit, *X*, with the log of aggregate consumption, $c = \log(C)$, and aggregate consumption follows a geometric Brownian motion. For expositional purposes, we presume, $\sigma_{xc} \geq 0$.

Let *P(X,B)* be the value of operating profit, *X*, before growth investments. Let *C(X,B)* be the cost of expected future expansion investments. In appendix A, we prove that the form of these value functions, *P(X,B)* and *C(X,B)* is,

$$P(X,B) = B\pi\left(Y\right) \text{ and } C(X,B) = B\chi\left(Y\right) \tag{5}$$

where $\pi(Y)$ and $\chi(Y)$ are functions of Y = ROC = X/B. Because $\pi(Y)$ and $\chi(Y)$ depend only on Y, the manager expands the business depending on Y and not its separate components (X and B). The manager expands the business when ROC exceeds a boundary that we denote as, ξ. Also in appendix A, we show that for this arbitrary expansion boundary, ξ, the value to capital ratio, $\frac{V(X,B)}{B} = \pi(Y) - \chi(Y)$, is,

$$\pi(Y)-\chi(Y)=\begin{cases} \frac{Y}{r^*}+\frac{g\xi}{r^*(r^*-g)}\frac{1-\lambda}{\alpha-\lambda}\left(Y/\xi\right)^{\alpha}-\frac{g}{r-g}\frac{\lambda}{(\lambda-\alpha)}\left(Y/\xi\right)^{\alpha}, & \textit{no growth},\ Y<\xi \\ \frac{Y}{r^*-g}+\frac{g\xi}{r^*(r^*-g)}\frac{1-\alpha}{\alpha-\lambda}\left(Y/\xi\right)^{\lambda}-\frac{g}{r-g}\left(1-\frac{\alpha}{(\alpha-\lambda)}\left(Y/\xi\right)^{\lambda}\right), & \textit{growth},\ \mathrm{Y}\geq\xi \end{cases} \tag{6}$$

where, $r^* \equiv r + \theta\sigma_{xc}$ and we define the constants $\alpha \geq 1$, $\lambda \leq 0$ in equation (A6).

On the "no growth" branch of equation (6), the first term is the value of operating profit if the manager *never* expands the business. This amount, which does not depend on the parameter "g," is the discounted value of expected future profit at the risk-adjusted rate r*. The second term (positive) and third term (negative) are respectively the value of expected future incremental profit generated and the cost of expansion investments incurred when, at some time in the future, ROC exceeds the expansion boundary, ξ, and the manager expands the business. The positive value of future incremental profit becomes more positive and the negative value for the cost of future expansion investment becomes more negative with ROC=Y because both incremental profit and expansion investment become more likely.
On the "growth" branch of equation (6), the first term is the value of operating profit if the manager *permanently* expands the business. Because the manager grows the business, this

term - for a growing perpetuity - recognizes the growth parameter, "g." The second term (negative) is the value of expected future profit forgone when ROC declines below the expansion boundary and the manager defers investment until ROC once more exceeds the expansion boundary, ξ. This term becomes less negative with ROC=Y because the possibility of forgone profit decreases with greater profitability. The third term (negative) is the cost of expansion expenditures, which recognizes that the manager avoids these costs when he/she defers investment for ROC below the expansion boundary. This term becomes more negative with ROC=Y because the likelihood of continued growth investment increases with profitability.

B. Value Maximizing Expansion boundary

To find the value maximizing expansion boundary, ξ^*, take the derivative of the value function, $\pi(Y)-\chi(Y)$, with respect to ξ on either branch of equation (6), set the result to zero, evaluate at Y=ξ, and solve for ξ^*,

$$\xi^* = r^* \times \left[\frac{r^*-g}{r-g}\right] \times \left[\frac{\alpha\lambda}{(1-\alpha)(1-\lambda)}\right] \tag{7}$$

The first two terms on the right hand side of equation (7) equal the manager's critical expansion ROC for a hypothetically *permanent* expansion decision. In this case, the value of the permanently growing firm, $\frac{X_0}{r^*-g}-\frac{gB}{r-g}$, exceeds the value of the permanently non-growing firm, $\frac{X_0}{r^*}$, when $Y=\text{ROC}\geq r^*\times\left(\frac{r^*-g}{r-g}\right)$. The term $\left(\frac{r^*-g}{r-g}\right)\geq 1$ represents a leverage "risk-cost" generated by expansion expenditures. The final term in (7) represents the impact of the manager's dynamic expansion ability on the value maximizing ROC threshold. Use the definitions of α and λ in equation (A6), to verify that a sufficient condition for this final term in equation (7) to be less than one is positive covariance risk, $\theta\sigma_{xc} \geq 0$. Because growth leverage risk is, at least in part, avoided by optional growth, the optional expansion boundary, ξ^*, is less than the permanent expansion boundary, $r^* \times \left(\frac{r^*-g}{r-g}\right)$. Further, with reasonable parameter values, the dynamic expansion boundary, ξ^*, is significantly less than the permanent expansion boundary, $r^* \times \left(\frac{r^*-g}{r-g}\right)$. For example, with parameters values that we use in figure 1, the permanent-expansion boundary is 54%, whereas, the dynamic expansion boundary is ξ* = 11.82%. The permanent expansion boundary is very large because not only do the "fixed costs" of growth impose risk on financial asset holders, but also, these costs *grow* unavoidably over time.

C. The Expansion Boundary and the Performance Hurdle Rate

In appendix B, we prove that the market to capital ratio equals one when ROC is at the expansion boundary, $\left.\frac{V(X,B)}{B}\right|_{Y=\xi^*} = \pi(\xi^*) - \chi(\xi^*) = 1$. This result means that investors' benchmark for corporate performance is the same as the manager's value maximizing benchmark for growth, ξ^*. Because the value to expenditure ratio, $\pi(Y) - \chi(Y)$, is increasing in ROC =Y, if Y ≥ ξ^*, then manager expands the business at the maximum rate *g* and, at the same time, the firm creates positive net value for financial asset holders because value exceeds capital $V(X,B) \geq B$ and $\pi(Y) - \chi(Y) \geq 1$ when Y ≥ ξ^*. Note that if the manager investments at any ROC less than the optimal expansion boundary, ξ^*, net value creation is negative.

D. The Cost of Capital

At least since Modigiliani and Miller (1958) Proposition I, academics and practitioners of corporate finance have measured a firm's cost of capital as the expected rate of return on the value of assets. In our case, this return is operating profit, *X*, less expansion expenditures (if incurred), plus expected capital gain from changes in operating profit, all divided by asset value. Define $\omega(Y)$ as the cost of capital. Using Ito's lemma, the cost of capital is,

$$\omega(Y) = \begin{cases} \dfrac{Y + \frac{1}{2}\frac{\partial^2 \psi_{NG}}{\partial Y^2}\sigma^2 Y^2}{\psi_{NG}}, & no\ \ growth,\ Y < \xi \\ \\ \dfrac{Y - g + \frac{1}{2}\frac{\partial^2 \psi_G}{\partial Y^2}\sigma^2 Y^2 + g\psi_G}{\psi_G}, & growth,\ \xi \le Y \end{cases} \tag{8}$$

where, $\psi_G(Y)$ and $\psi_{NG}(Y)$ are the market to capital ratios, $\frac{V(X,B)}{B} = \pi(Y) - \chi(Y)$, in the growth and no-growth states, respectively, as given by equation (6). In appendix C, we prove the assertions that we make in the following two paragraphs. Figure 1 depicts these results for a numerical example.

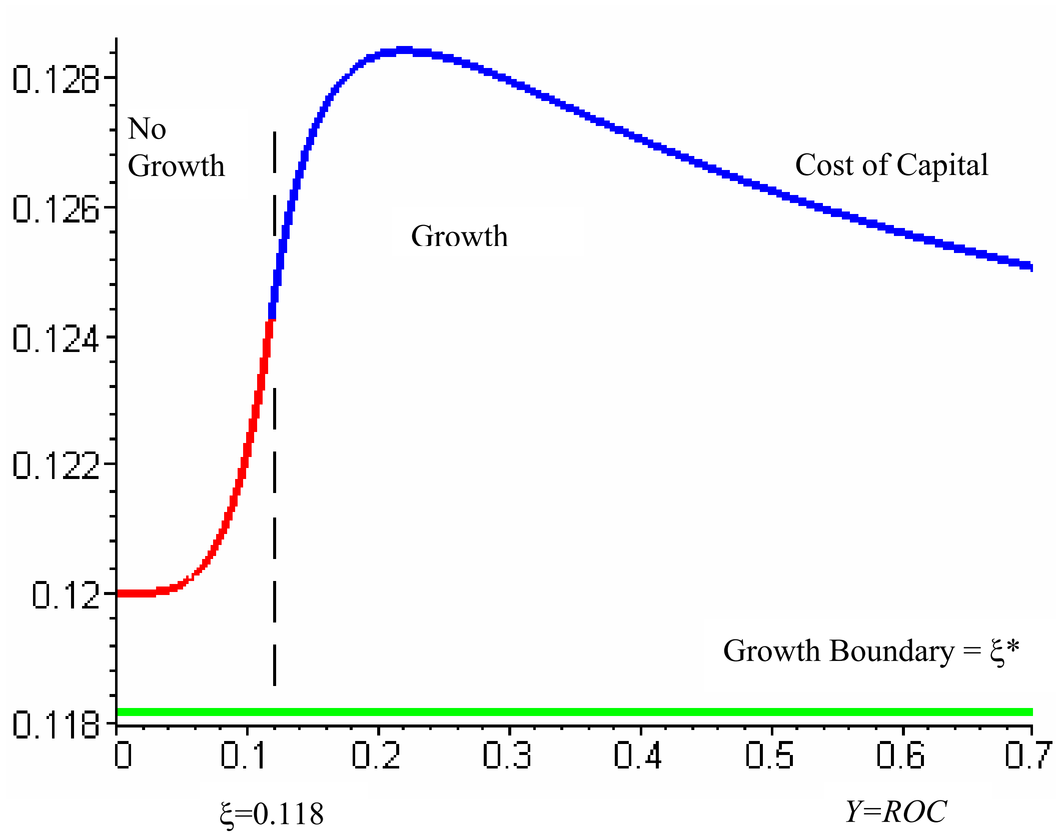
Unlike Modigliani and Miller (1958), the cost of capital, $\omega(Y)$, changes with profitability, *Y*, in a dynamic environment. This dependence represents the changing prospects of incurring growth investments and leverage. Of course, Modigliani and Miller (1958) do not consider a dynamic economic environment, and therefore, they do not identify these altering prospects. As the return on capital, *Y*, increases from zero, growth leverage, and thus, the cost of capital, $\omega(Y)$, increases initially and then decreases. Leverage risk increases, as ROC increases from zero, where the business is in the no-growth state, because of increasing likelihood that ROC will exceed the expansion boundary, ξ^*, and the manager invests for growth, which incurs growth expenditures. As the return on capital, *Y*, approaches its lower bound of zero, the likelihood of an increase back to the expansion boundary, ξ^*, is remote. With no likelihood of incurring capital expenditures for growth, the "risk cost" of growth leverage disappears and the cost of capital reflects only the "risk cost" of in place operations. As ROC falls to zero from the right, $\lim_{Y \to 0+} \omega(Y) = r + \theta\sigma_{x,c}$. The cost of capital, $\omega(Y)$, eventually decreases as profitability increases and the business is better able to cover growing capital expenditures required for continued profit growth. As ROC increases without limit, $\lim_{Y \to \infty} \omega(Y) = r + \theta\sigma_{x,c}$.

The cost of capital, $\omega(Y)$, increases as ROC increases through the expansion boundary, ξ^*, before the cost of capital, $\omega(Y)$, reaches its maximum. Further, $\omega(Y)$ has an inflection point at the expansion boundary, $Y = \xi^*$. The change in the cost of capital, $\frac{d\omega(Y)}{dY}$, is greatest at this point. Both these observations suggest that when ROC is at the expansion boundary, $Y = \xi^*$, the fraction of future time that the business will be in the growth state is modest. ROC must be greater than the expansion boundary, $Y > \xi^*$, before the likelihood of remaining in the growth state is significant. Only then, does the cost of capital reach its maximum (at approximately *Y*=20% in figure 1). At this ROC, above the expansion boundary, the likelihood of regularly incurring expansion expenditures is great enough so that the "risk cost" of leverage is at a maximum. The ROC that maximizes the cost of capital, $\omega(Y)$, is greater than the expansion boundary, ξ^*.

Figure 1:

Numerical Example for the Cost of Capital and the Expansion Boundary

$g = 0.03,\ r = 0.05,\ \theta\sigma_{x,c} = 0.07\ ,\ \sigma = 0.2$



A common view in the managerial practice of corporate finance, which most text books on corporate finance represent, is that an investment that expands existing operating profit, but otherwise is of the same character as in place operations, is of the same risk as in place operations. This commonly held view is erroneous. Our results indicate that even if expansion investments are identical scaled copies of existing operations, they are of greater risk. The cost of capital for the existing business – a hypothetically non-growing business – is the minimum cost of capital, $r+\theta\sigma_{x,c}$. On the other hand, because the cost of capital, $\omega(Y)$, which reflects the risk of the existing business plus growth opportunities, exceeds the minimum cost of capital, $\omega(Y)\geq r+\theta\sigma_{X,C}$, growth investments are riskier than the existing business. Because growth is not spontaneous, the manager must invest to create profit growth. This investment creates growth leverage risk that does not exist for in place operations. The cost of capital for a firm with expansion options always exceeds that of a firm with no expansion options.

Even with optional expansion, financial asset holders are not immune to growth leverage. The numerical example of figure 1 reflects a rather modest fall in the cost of capital, $\omega(Y)$, with increasing profitability, *Y*, for large profitability. As ROC increases from approximately 20% (above the expansion boundary and near the maximum cost of capital) to 70% (well above the expansion boundary) the risk premium falls only (approximately) from 12.8%-5%=7.8% to 12.6%-5%=7.6%. Because profit growth requires capital growth (increasing fixed costs), leverage "risk-cost" is insensitive to profitability in this range. ROC must be very great to "cover" increasing fixed costs and eliminate growth "risk costs." Possibly this decrease in the cost of capital is modest because, while 70% seems a high return, it is only approximately 3 standard deviations above the value maximizing expansion boundary in figure 1. Further, because the logarithm of ROC follows an arithmetic Brownian motion, despite the fact that ROC is very high, 70%, ROC falls below the expansion boundary at some time in the future with probability one (see Cox and Miller (1965)).

E. Comparing the Expansion Boundary and the Cost of Capital

In appendix C, we prove that the minimum cost of capital, $r^* = r + \theta\sigma_{x,c}$, exceeds the value maximizing ROC benchmark for growth, ξ^*. Consequently, the cost of capital, $\omega(Y)$, exceeds the growth boundary, $\omega(Y) \geq \xi^*$. This result establishes that the value maximizing expansion boundary is uniformly less than the cost of capital. Consequently, practical application of the cost of capital for business expansion analysis or for corporate performance assessment is overly conservative. The manager undertakes an expansion today that appears marginal, that is, ROC less than the cost of capital, because this investment enhances capacity to capture greater future value with large investments when business earnings improve. Marginal expansion investments create the facilities and competence that a small firm needs to capture greater value with large and more profitable future investments.

In figure 1, the cost of capital exceeds the expansion boundary by approximately 20 basis points when ROC=0% and by over 100 basis points when $\text{ROC} \approx 22\%$. When the manager expands the business, $\text{ROC} \geq 11.8\%$, but ROC is less that the right most boundary in figure 1, ROC = 70%, the cost of capital exceeds the expansion boundary by at least 60 basis points. These differences illustrate that managers that use the cost of capital for growth analysis and investors that use the cost of capital for corporate performance measurement risk grave decision errors.

One might mistakenly argue that the cost of capital uniformly exceeds the dynamic expansion boundary because of an option to defer non-growth when the business is in the growth state. A low expansion boundary defers non-growth relative to growth. This argument is erroneous because there is no reason to defer non-growth for growth when we presume in our model of business expansion no cost to enter the no-growth state from the growth state. Rather, the appropriate interpretation of our result is that McDonald and Siegel's (1986) and Dixit and Pindyck's (1994) option to defer investment for non-investment because of investment irreversibility – in our case, growth investment – is dominated – at least in the case of the proportional growth investments that we investigate in this paper – by another option: the option to invest currently to increase the value of future growth investments when business profits stochastically improve.

In the following four subsections of this paper and in the four panels of figure 2, we present a comparative static numerical analysis of the value maximizing expansion boundary, ξ^*, the minimum cost of capital, $r^* = r + \theta\sigma_{x,c}$, and the *maximum* cost of capital, $\omega^* \equiv \max_{0<Y<\infty} \omega(Y)$.

F. Cost of Capital and the Expansion Boundary Versus Volatility

The expansion boundary, ξ^*, decreases with volatility, σ, (see panel A of figure 2). The proportionality of growth option value with firm size enhances the appeal of an investment's upside earnings potential to a manager. This appeal reduces the dynamic expansion boundary, ξ^*, below the minimum cost of capital, r^*. This appeal is greater for greater earnings volatility, σ, and therefore, the dynamic expansion boundary, ξ^*, decreases with volatility. Note, in the left most section of panel A of figure 2, that when earnings volatility, σ, approaches zero, the dynamic expansion boundary approaches the minimum cost of capital, r^*.

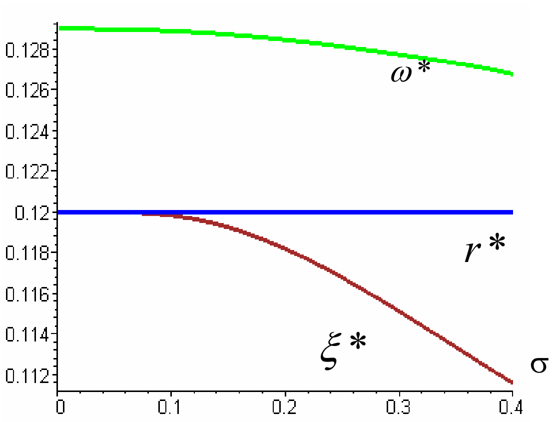
The comparative static result above is opposite to the investment deferral option for a solitary investment in Dixit and Pindyck (1994). Downside earnings risk for a one-time irreversible investment is an essential feature of Dixit and Pindyck's (1994) study of investment under uncertainty and the investment deferral option. With greater earnings volatility, this downside risk is greater and the profit boundary to start a solitary the investment increases because the investment becomes less attractive to the manager.

When profit volatility approaches zero, $\sigma \to 0$, the maximum cost of capital, ω^*, exceeds the expansion boundary, ξ^*, by almost 90 basis points. If profit volatility is greater, $\sigma \to 0.40$, then the maximum cost of capital, ω^*, exceeds the expansion boundary, ξ^*, by about 150 basis points. This divergence between the maximum cost of capital, ω^*, and the expansion boundary, ξ^*, illustrates that the maximum cost of capital, ω^* (and by extension, the cost of capital, $\omega(Y)$, itself) is insensitive to volatility compared to the growth boundary, ξ^*. The maximum cost of capital, ω^*, decreases, though modestly, with volatility, σ, because the expansion option's value increases with volatility, which increases business value, which decreases the cost of capital.

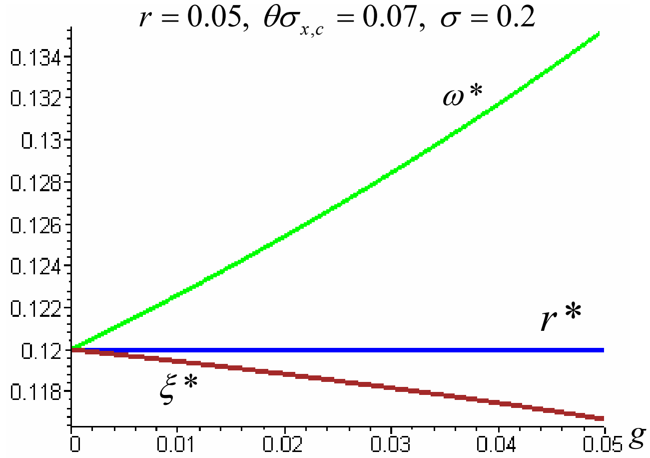
Figure 2
Comparative Statics:

Maximum Cost of Capital, Minimum Cost of Capital, and the Expansion Boundary,
$\omega^* \equiv \max_{0<Y<\infty} \omega(Y)$, $r^* = r + \theta\sigma_{xc}$, **and** ξ^*.

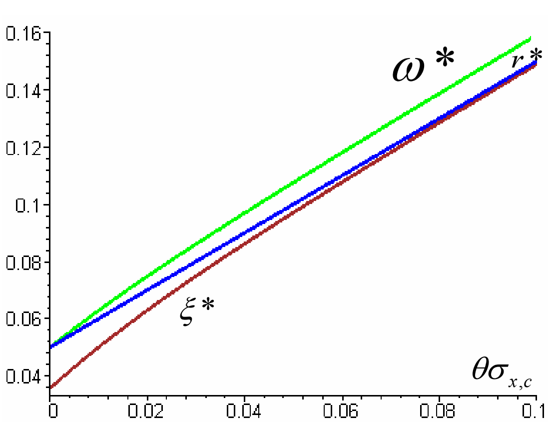
Panel A
$g = 0.03,\ r = 0.05,\ \theta\sigma_{x,c} = 0.07$



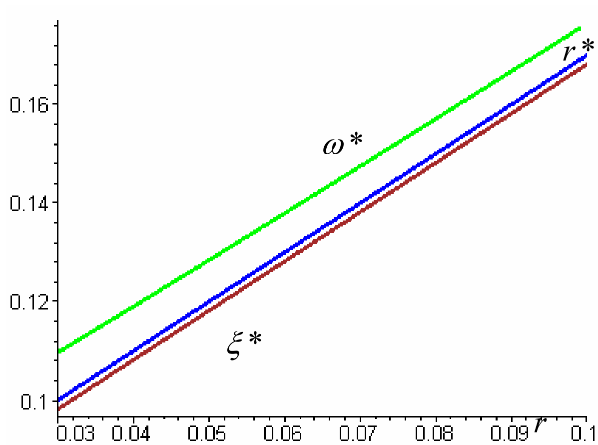
Panel B
$r = 0.05,\ \theta\sigma_{x,c} = 0.07,\ \sigma = 0.2$



Panel C
$g = 0.03,\ r = 0.05,\ \sigma = 0.20$



Panel D
$\theta\sigma_{x,c} = 0.07,\ \sigma = 0.2,\ g = 0.03$



G. Cost of Capital and the Expansion Boundary Versus Maximum Growth

In panel B of figure 2, the maximum cost of capital, ω^*, increases with maximal growth, *g*, because of increasing growth leverage. On the other hand, the expansion boundary, ξ^*, decreases modestly with maximal growth, *g*. Greater growth, *g*, enhances the appeal of upside earnings potential to the manager, who therefore, reduces the expansion boundary, ξ^*. However, the optimal growth boundary, ξ^*, is relatively insensitive to the rate of growth, *g*, because the benefit of expansion, profit growth, *g*, matches rather closely the burden of expansion, capital growth, *g*. Maximal growth, *g*, has a more pronounced impact on the cost of capital, ω^*, than it does on the expansion boundary, ξ^*. Maximum growth, *g*, is a primary determinant of the cost of capital, but not the value maximizing expansion boundary.

H. Cost of Capital and the Expansion Boundary Versus Covariance Risk

In panel C of figure 2, covariance, $\theta\sigma_{x,c}$, increases both the cost of capital, ω^*, and the expansion boundary, ξ^*. Covariance risk makes operating profit less attractive to the manager, who therefore, increases the expansion boundary, ξ^*, as covariance risk, $\theta\sigma_{x,c}$, increases.

Note that when covariance risk is zero, $\theta\sigma_{x,c} = 0$, both the maximum cost of capital and the minimum cost of capital equal the riskless rate, $\omega^* = r$ and $r^* = $ r. However, the expansion boundary, ξ^*, is less than the riskless interest rate ($\xi^* < r$, when $\theta\sigma_{x,c}$=0). The difference between the riskless interest rate and the expansion boundary, $r - \xi^*$, is about 150 basis points for zero covariance risk, $\theta\sigma_{x,c}$=0 in panel C of figure 2.

Measure the impact of growing growth options on a manager's incentive to make expansion investments with the difference between the maximum cost of capital and the expansion boundary, $\omega^* - \xi^*$. Panel C of figure 2 illustrates that this incentive is invariant to covariance risk, $\theta\sigma_{x,c}$. Note that even though the expansion boundary approaches the minimum cost of capital as covariance risk becomes large, $\xi^* \to r^*$ when $\theta\sigma_{x,c} \to 0.1$ in panel C of figure 2, the difference between the maximum cost of capital and the expansion boundary, $\omega^* - \xi^*$, is always about 150 basis points. Greater covariance increases the discounting of future growth investments, which reduces the incentive to make expansion investments today. However, at the same time, increasing covariance increases the cost of capital, and therefore, covariance risk has little impact on the difference between the maximum cost of capital and the expansion boundary, $\omega^* - \xi^*$.

I. The Cost of Capital and the Expansion Boundary Versus Riskless Interest Rates

In panel D of figure 2, as one would expect, interest rates increase the maximum cost of capital, ω^*. In addition, because greater interest discourages investment, greater interest increases the expansion boundary, ξ^*. The riskless rate of interest, *r,* has little influence on the difference between the expansion boundary and the maximum cost of capital, $\omega^* - \xi^*$. This difference is approximately 120 basis points in panel D of figure 2.

J. Primary Determinants of the Cost of Capital and the Expansion Boundary

The panels of figure 2 illustrate that the parameters of business value that determine the minimum cost of capital, $r + \theta\sigma_{x,c}$, influence the cost of capital and the expansion boundary in a like manner. In panels C and D, the maximum cost of capital, ω^*, and the expansion boundary, ξ^*, both increase in parallel with the riskless rate, *r*, and covariance risk, $\theta\sigma_{x,c}$.

In panel B of Figure 2, growth, *g*, has a modest negative effect on the growth boundary, ξ^*, whereas, it has a positive impact, through growth leverage, on the cost of capital, ω^*. Growth, g, has little impact on the expansion boundary, ξ^*, because the benefit of expansion - profit growth – closely matches the cost of expansion – capital investment. On the other hand, a manager's discretionary expansion expenditures impose a leverage risk on financial asset holders. The cost of capital reflects this risk, and therefore, the cost of capital, ω^*, increases with maximal growth, *g*.

In panel A of figure 2, volatility, σ, has a modest negative effect on the cost of capital through the expansion option's value, whereas, volatility, σ, has a negative impact on the expansion boundary, ξ^*. Volatility has little impact on the cost of capital because investors anticipate the influence of volatility on the manager's expansion decision. Like in many asset-pricing models, volatility is largely unpriced in equilibrium because its influence is diminished by portfolio diversification. The proportionality of growth option value with firm size increases the appeal of an investment's upside earnings potential to a manager compared to a sequence of same sized growth investments. Greater earnings volatility further enhances this appeal, which induces the manager to reduce the expansion boundary, ξ^*.

These observations indicate that the primary determinants of the cost of capital are: riskless interest rates, covariance risk, and growth leverage. The primary determinants of the optimal expansion boundary are: riskless interest rates, covariance risk, and profit volatility. The primary determinants of the dynamic expansion boundary and the cost of capital differ.

These differences are important for framing empirical tests of the dynamic business expansion model that we propose in this paper.

III. Summary and Conclusion

Our main result in this paper is that the cost of capital is inadequate for either investment decisions or performance evaluation when a firm faces a dynamic opportunity to expand with investments that are proportional to existing capital. Instead, we derive an investment boundary that maximizes firm value.

A manager must calculate the expansion boundary as the cost of capital *less* two positive amounts. The first amount undoes the effect of growth leverage. This positive factor is the difference between the uppermost line and the middle line ($\omega^* - r^*$) in any of the four panels of figure 2. Second, the manager must subtract a second positive amount from the cost of capital to recognize the appeal of a firm's growing growth options. This positive factor is the difference between the middle line and the lowermost line ($r^* - \xi^*$) in any of the four panels of figure 2.

For reasonable parameter values, our investment boundary can differ from the cost of capital by up to 150 basis points (see, for example, panel C or panel D of figure 2). Furthermore, this difference is not uniform in firm characteristics. For example, because the difference between the cost of capital and the value maximizing expansion boundary tends to increase with earnings volatility (see panel A of figure 2), the manager of a firm with highly volatile operating earnings, who inappropriately uses the cost of capital for expansion analysis, will, more often, reject value creating expansion investments.

Appendix A:

The Value of Operating Profit

With a constant riskless interest rate, *r*, the value function, *P(X,B)*, satisfies the differential equation, $rPdt = Xdt + E(dP)$. Use the branches for dB/B from equation (2), apply Ito's Lemma to *dP*, and with the risk adjusted process for operating profit in equation (4):

$$rP = \begin{cases} X - \theta\sigma_{xc} X P_X + \frac{\sigma^2}{2} X^2 P_{XX}, & no\ \ growth,\ \ X < \xi(B) \\ X + (g - \theta\sigma_{xc}) X P_X + \frac{\sigma^2}{2} X^2 P_{XX} + gBP_B, & growth,\ X \geq \xi(B) \end{cases} \tag{A1}$$

We conjecture (and verify) that the value function *P(X,B)* is of the form:

$$P(X,B) = B\pi\left(Y\right) \tag{A2}$$

where *π(Y)* is a function of *Y* = ROC. Note that:

$$\begin{aligned} \frac{\partial P}{\partial X} &= B\pi' \frac{1}{B} = \pi' \\ \frac{\partial^2 P}{\partial X^2} &= \frac{\partial \pi'}{\partial X} = \pi'' \frac{1}{B} \\ \frac{\partial P}{\partial B} &= \pi - B\pi' \frac{X}{B^2} = \pi - \frac{X}{B}\pi' \end{aligned} \tag{A3}$$

Substitute (A3) into (A1) and after dividing both sides by *B*:

$$r\pi = \begin{cases} Y - \theta\sigma_{xc} Y\pi' + \frac{\sigma^2}{2} Y^2 \pi'', & no\ \ growth,\ \ Y < \xi \\ Y - \theta\sigma_{xc} Y\pi' + \frac{\sigma^2}{2} Y^2 \pi'' + g\pi, & growth,\ \ Y \geq \xi \end{cases} \tag{A4}$$

For, $0 \leq g < r^* \equiv r + \theta\sigma_{xc}$, the solutions to these ordinary differential equations are:

$$\pi(Y)=\begin{cases}\dfrac{Y}{r^*}+C_1Y^{\alpha}, & no\ growth,\ Y<\xi\\ \\ \dfrac{Y}{r^*-g}+C_2Y^{\lambda}, & growth,\ Y\geq\xi,\end{cases} \tag{A5}$$

$$\alpha\equiv\frac{1}{2}+\frac{\theta\sigma_{xc}}{\sigma^2}+\sqrt{\frac{2r}{\sigma^2}+\left(\frac{1}{2}+\frac{\theta\sigma_{xc}}{\sigma^2}\right)^2}\geq 1,$$

$$\lambda\equiv\frac{1}{2}+\frac{\theta\sigma_{xc}}{\sigma^2}-\sqrt{\frac{2(r-g)}{\sigma^2}+\left(\frac{1}{2}+\frac{\theta\sigma_{xc}}{\sigma^2}\right)^2}\leq 0, \tag{A6}$$

Determine the parameters C_1 and C_2 with value matching and smooth pasting conditions at $Y = \xi$ (see, Dixit and Pyndyck (1994) for a discussion of these conditions). Solve these two equations (not given),

$$\pi(Y)=\begin{cases}\dfrac{Y}{r^*}+\dfrac{g\xi}{r^*(r^*-g)}\dfrac{1-\lambda}{\alpha-\lambda}\left(Y/\xi\right)^{\alpha}, & no\ growth,\ Y<\xi\\ \\ \dfrac{Y}{r^*-g}+\dfrac{g\xi}{r^*(r^*-g)}\dfrac{1-\alpha}{\alpha-\lambda}\left(Y/\xi\right)^{\lambda}, & growth,\ Y\geq\xi\end{cases} \tag{A7}$$

The Cost of Future Expansion investments

Let *C(X,B)* be the cost of expected future expansion investments. Loosely speaking, *C(X,B)* is the discounted value of expected future expansion expenditures. When expanding the business, the manager incurs investment expenditures at the instantaneous rate $gBdt$. The function, *C(X,B)*, satisfies the pair of differential equations,

$$rCdt=\begin{cases}E(dC), & no\ growth,\ X<\xi(B)\\ \\ gBdt+E(dC), & growth,\ X\geq\xi(B)\end{cases} \tag{A8}$$

Apply Ito's Lemma, and with the risk adjusted process for operating profit in (4),

$$rC = \begin{cases} -\theta\sigma_{xc} X C_X + \frac{\sigma^2}{2} X^2 C_{XX}, & no\ \ growth, X < \xi(B) \\ \\ gB + (g - \theta\sigma_{xc}) X C_X + \frac{\sigma^2}{2} X^2 C_{XX} + gBC_B, & growth,\ X \geq \xi(B) \end{cases} \tag{A9}$$

Like the previous Subsection, we conjecture (and verify, but details are not given) that *C(X,B)* is of the form:

$$C(X,B) = B\chi\left(Y\right) \tag{A10}$$

For, $0 \leq g < r$, the solutions to the ordinary differential equations in (A9), with value-matching and smooth-pasting conditions at $Y=\xi$, are,

$$\chi(Y) = \begin{cases} \frac{g}{r-g}\frac{\lambda}{(\lambda-\alpha)}\left(Y/\xi\right)^{\alpha}, & no\ \ growth,\ Y < \xi \\ \\ \frac{g}{r-g}\left(1 - \frac{\alpha}{(\alpha-\lambda)}\left(Y/\xi\right)^{\lambda}\right), & growth,\ Y \geq \xi \end{cases} \tag{A11}$$

The Value Function

Asset value is the value of operating profits less the cost of expected future expansion investments, $V(X,B) \equiv P(X,B) - C(X,B) = B\left(\pi - \chi\right)$. Equation (6) gives the value to expenditure ratio, $\pi - \chi$.

Appendix B:

In this appendix we prove $\left.\frac{V(X,B)}{B}\right|_{Y=\xi^*} = \pi(\xi^*) - \chi(\xi^*) = 1$. Substitute the expression for ξ^* on the right hand side of (7) into either branch of (6),

$$\left.\frac{V(X,B)}{B}\right|_{Y=\xi^*} = \pi(\xi^*) - \chi(\xi^*) = \frac{-\lambda\left(-r^*\alpha^2 + \alpha r^*\lambda + g\alpha^2 - \alpha g\lambda - g + g\lambda\right)}{(r-g)(\alpha-1)(\lambda-1)(\alpha-\lambda)}$$

Substitute definitions for α, λ, and r^* from (A6), into the above equation. To avoid a great deal of algebra, use a symbolic mathematical computer program, like, MAPLE® or MATHEMATICA® to verify that $\left.\frac{V(X,B)}{B}\right|_{Y=\xi^*} = \pi(\xi^*) - \chi(\xi^*) = 1$.

Appendix C

This appendix proves that the optimal expansion boundary, ξ^*, is always below the cost of capital, $\omega(Y)$. We presume positive covariance risk, $\sigma_{xc} \geq 0$. Simplify (8):

$$\omega(Y) = \begin{cases} \dfrac{1 + \frac{1}{2}\gamma(\alpha-1)\sigma^2 Y^{\alpha-1}}{\frac{1}{r^*} + \frac{\gamma}{\alpha} Y^{\alpha-1}}, & \textit{no growth, } Y < \xi \\[2em] \dfrac{Y - g - \frac{\delta\lambda\sigma^2 Y^{\lambda}}{2}}{\frac{Y}{r^*-g} - \frac{g}{r-g} + \frac{\delta Y^{\lambda}}{1-\lambda}} + g, & \textit{growth, } \xi \leq Y \end{cases} \tag{C1}$$

where $\gamma = \frac{g}{r^*(r^*-g)}\frac{1-\lambda}{\alpha-\lambda}\frac{1}{\xi^{\alpha-1}} > 0$ and $\delta = \frac{g}{r-g}\frac{\alpha}{\alpha-\lambda}\frac{1}{\xi^{\lambda}} > 0$.

We show the following steps:

1. The minimum cost of capital is r^*, $\min_{0 \leq Y \leq \infty} \omega(Y) = r^*$.

    a. The cost of capital equals the risk-adjusted rate for $Y = 0$, $\omega(0) = r^*$.

    b. The cost of capital approaches r^* as Y increases without bound, $\lim_{Y\to\infty} \omega(Y) = r^*$.

c. On the no-growth branch, $\omega(Y) \geq r^*$, for $0 \leq Y \leq \xi$, because the cost of capital, equation (C1), is an increasing function of *Y*.

d. On the growth branch of equation (C1), $\omega(Y) \geq r^*$ because $\omega(Y)$ has only one extreme point, which is a maximum, for $Y \geq \xi$.

2. The expansion boundary never exceeds the minimum cost of capital, $\xi^* \leq r^*$.

Therefore, the cost of capital exceeds the expansion boundary, $\omega(Y) \geq \xi^*$, for $Y \geq 0$.

1. The minimum cost of capital is r^*.

a. The cost of capital equals the risk-adjusted rate at $Y = 0$. On the no-growth branch, the cost of capital is:

$$\omega(Y) = \frac{1 + \frac{1}{2}\gamma(\alpha - 1)\sigma^2 Y^{\alpha-1}}{\frac{1}{r^*} + \frac{\gamma}{\alpha}Y^{\alpha-1}} \tag{C2}$$

At $Y = 0$, the cost of capital simplifies to $\omega(0) = r^*$.

b. The cost of capital approaches r^* as *Y* increases without bound, $\lim_{Y \to \infty} \omega(Y) = r^*$.

On the growth branch, the cost of capital is:

$$\omega(Y) = \frac{Y - g + \frac{1}{2}\frac{\partial^2 \Psi_G}{\partial Y^2}\sigma^2 Y^2 + g\Psi_G}{\Psi_G} \tag{C3}$$

The denominator of (C3) tends to infinity:

$$\lim_{Y \to \infty} \Psi_G = \lim_{Y \to \infty}\left(\frac{Y}{r^* - g} - \frac{g}{r - g}\right) = \infty \tag{C4}$$

Therefore, the limit of the cost of capital is:

$$\lim_{Y\to\infty}\omega(Y)=\lim_{Y\to\infty}\left(\frac{Y}{\Psi_G}-\frac{g}{\Psi_G}+\frac{\frac{1}{2}\frac{\partial^2\Psi_G}{\partial Y^2}\sigma^2Y^2}{\Psi_G}+g\right) \tag{C5}$$

The first term of (C5) tends to r^* - g, the second term tends to zero, and the third term is:

$$\lim_{Y\to\infty}\omega(Y)=\lim_{Y\to\infty}\frac{\frac{1}{2}\frac{\partial^2\Psi_G}{\partial Y^2}\sigma^2Y^2}{\Psi_G}=\lim_{Y\to\infty}\left(\frac{1}{2}\frac{\partial^2\Psi_G}{\partial Y^2}\sigma^2Y(r^*-g)\right)=$$
$$=\lim_{Y\to\infty}\left(\frac{1}{2}\sigma^2(r^*-g)\left(\frac{g\xi}{r^*(r^*-g)}\frac{1-\alpha}{\alpha-\lambda}-\frac{g}{r-g}\frac{\alpha}{\alpha-\lambda}\right)\frac{\lambda^2Y^{\lambda-1}}{\xi^\lambda}\right)=0 \tag{C6}$$

Therefore,

$$\lim_{Y\to\infty}\omega(Y)=r^* \tag{C7}$$

c. The no-growth branch of the cost of capital, equation (C1), is an increasing function of Y.

Differentiate the no-growth branch of Equation (C1) with respect to Y:

$$\frac{\partial\omega}{\partial Y}=\frac{\left(\frac{1}{2}\gamma(\alpha-1)^2\sigma^2Y^{\alpha-2}\right)\left(\frac{1}{r^*}+\frac{\gamma}{\alpha}Y^{\alpha-1}\right)-\left(1+\frac{1}{2}\gamma(\alpha-1)\sigma^2Y^{\alpha-1}\right)\left(\frac{\gamma(\alpha-1)}{\alpha}Y^{\alpha-2}\right)}{\left(\frac{1}{r^*}+\frac{\gamma}{\alpha}Y^{\alpha-1}\right)^2} \tag{C8}$$

This derivative is positive because:

$$\frac{\sigma^2(\alpha-1)}{2}\left(\frac{1}{r^*}+\frac{\gamma}{\alpha}Y^{\alpha-1}\right)\geq\frac{1}{\alpha}\left(1+\frac{1}{2}\gamma(\alpha-1)\sigma^2Y^{\alpha-1}\right) \tag{C9}$$

or,

$$\alpha(\alpha-1)\geq\frac{2r^*}{\sigma^2} \tag{C10}$$

Substitute for α and simplify to obtain:

$$\frac{2\theta\sigma_{xc}}{\sigma^2}+2\sqrt{\frac{2r}{\sigma^2}+\left(\frac{1}{2}+\frac{\theta\sigma_{xc}}{\sigma^2}\right)^2}\geq 1 \qquad \text{(C11)}$$

which holds because $2\sqrt{\frac{2r}{\sigma^2}+\left(\frac{1}{2}+\frac{\theta\sigma_{xc}}{\sigma^2}\right)^2} > 2\sqrt{\left(\frac{1}{2}+\frac{\theta\sigma_{xc}}{\sigma^2}\right)^2} = 1+\frac{2\theta\sigma_{xc}}{\sigma^2}\geq 1$.

d. The growth branch of the cost of capital, equation (C1), has only one extreme point, which is a maximum, for $Y > \xi$.

Differentiate the growth branch of equation (C1) with respect to Y:

$$\frac{\partial\omega}{\partial Y}=\frac{\frac{g}{r^*-g}-\frac{g}{r-g}+\delta\left(1-\frac{\lambda\sigma^2(1-\lambda)}{2(r^*-g)}\right)Y^{\lambda}+g\delta\lambda\left(\frac{\lambda\sigma^2}{2(r-g)}+\frac{1}{1-\lambda}\right)Y^{\lambda-1}}{\left(\frac{Y}{r^*-g}-\frac{g}{r-g}+\frac{\delta Y^{\lambda}}{1-\lambda}\right)^2} \qquad \text{(C12)}$$

The numerator of equation (C12) has at most 2 roots, for $Y > 0$. Note that by the smooth pasting condition, $\left.\frac{\partial\omega}{\partial Y}\right|_{Y=\xi} > 0$ on either branch of equation (C1). It is easily verified that the first derivative of the growth branch, equation (C12), is negative for small ROC, $\lim_{Y\to 0}\left(\frac{\partial\omega}{\partial Y}\right) < 0$[12]. Therefore, equation (C12) has one root for $Y \leq \xi$. Therefore, Equation (C12) has at most one root for $Y > \xi$. In other words, the cost of capital, equation (C1), has at most one extreme value for $Y > \xi$. Combined with parts 1(a), 1(b), and 1(c) of this proof, this result means that $\omega(Y) > r^*$.

To review, because the cost of capital, $\omega(Y)$, has at most one extreme point for $Y > \xi$, because $\omega(Y)$ is an increasing function at $Y = \xi$, is above r^* at $Y = \xi$, and approaches r^* as Y

[12] The third term of the numerator determines the sign of the derivative as Y approaches zero from the right. Substituting λ into the third term shows that this term is negative.

tends to infinity, the extreme point is a maximum. The cost of capital, $\omega(Y)$, therefore, exceeds *r** for any $Y \geq \xi$.

2. The expansion boundary never exceeds the minimum cost of capital, $\xi^* \leq r^*$.

A rearrangement of equation (7) reveals that $\xi^* \leq r^*$ when,

$$S \equiv \theta\sigma_{x,c}\alpha\lambda - (r-g)(1-\alpha-\lambda) \geq 0 \qquad \text{(C13)}$$

Consider *S* as a function of the parameter *g*. We show that $S(g) \geq 0$ for $0 \leq g \leq r$.

Substitute the definitions for α and λ and verify that $S(g)\big|_{g=0} = 0$ and $S(g)\big|_{g=r} = 0$. The first order condition for the maximum value of the function *S* with respect to *g*, is,

$$g = \theta\sigma_{x,c}\alpha + \sigma^2(1-\alpha-\lambda)\sqrt{\frac{2(r-g)}{\sigma^2} + \left(\frac{1}{2} + \frac{\theta\sigma_{xc}}{\sigma^2}\right)^2} + r \qquad \text{(C14)}$$

Denote the right hand side of equation (C14) by

$$h(g) \equiv \theta\sigma_{x,c}\alpha + \sigma^2(1-\alpha-\lambda)\sqrt{\frac{2(r-g)}{\sigma^2} + \left(\frac{1}{2} + \frac{\theta\sigma_{xc}}{\sigma^2}\right)^2} + r \qquad \text{(C15)}$$

The second derivative of *h*(*g*) is:

$$\frac{\partial^2 h}{\partial g^2} = \frac{1}{\left(\frac{2(r-g)}{\sigma^2} + \left(\frac{1}{2} + \frac{\theta}{\sigma^2}\right)^2\right)\sigma^2} + \frac{\frac{2\theta}{\sigma^2} + \sqrt{\frac{2r}{\sigma^2} + \left(\frac{1}{2} + \frac{\theta}{\sigma^2}\right)^2} - \sqrt{\frac{2(r-g)}{\sigma^2} + \left(\frac{1}{2} + \frac{\theta}{\sigma^2}\right)^2}}{\left(\sqrt{\frac{2r}{\sigma^2} + \left(\frac{1}{2} + \frac{\theta}{\sigma^2}\right)^2}\right)^3 \sigma^2} \qquad \text{(C16)}$$

which is clearly positive for $0 \leq g \leq r$. Therefore, the function *h*(*g*) is convex for $0 \leq g \leq r$. Further, it is easy to verified that $h(g)\big|_{g=0} \geq 0$ and $h(g)\big|_{g=r} \leq r$. Therefore, equation (C14) has exactly one solution for $0 \leq g \leq r$, which is the only extreme point of the function *S*(*g*).

The last step of the proof requires that we establish that S is positive rather than negative at this extreme point.

To verify that this extreme point is a maximum, not a minimum, take the derivative of *S* with respect to *g* and evaluate it at *g* = 0:

$$\left.\frac{\partial S}{\partial g}\right|_{g=0} = \theta\sigma_{x,c}\alpha + \sigma^2\left(1-2\left(\frac{1}{2}+\frac{\theta\sigma_{x,c}}{\sigma^2}\right)\right)\sqrt{\frac{2r}{\sigma^2}+\left(\frac{1}{2}+\frac{\theta\sigma_{x,c}}{\sigma^2}\right)^2}+r = \\ = \theta\sigma_{x,c}\alpha - 2\theta\sigma_{x,c}\sqrt{\frac{2r}{\sigma^2}+\left(\frac{1}{2}+\frac{\theta\sigma_{x,c}}{\sigma^2}\right)^2}+r \qquad \text{(C17)}$$

Simplification shows that $\left.\frac{\partial S}{\partial g}\right|_{g=0} \geq 0$ if $r(r+\theta\sigma_{x,c}) \geq 0$ which clearly holds.

Because the function *S*(*g*) is increasing at *g* = 0 and has only one extreme value for $0 \leq g \leq r$, this extreme value must be a maximum. Therefore, *S*(*g*) is non-negative, $S(g) \geq 0$ for $0 \leq g \leq r$.

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